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Geometric pattern infill influence on pentagonal cupola mechanical behavior subject to static external loads

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Abstract. Every day we are surrounded, and we enter in contact with tens or maybe even hundreds of objects, whether they serve us as tools in carrying out certain activities, or whether they have a purely decorative role, they are part of our lives and have become indispensable to humanity. But how many times have we thought about the fact that these objects are based on simple geometries or intersections of geometric bodies that together make up an assembly which, later can be built and used for its designed purpose. This article aims to analyse the mechanical behaviour of the pentagonal cupola when applying static loads, considering several geometric filling patterns, but also the empty and the full structure of the polyhedron.

Keywords: pentagonal cupola, Johnson solids, static simulation, geometric pattern

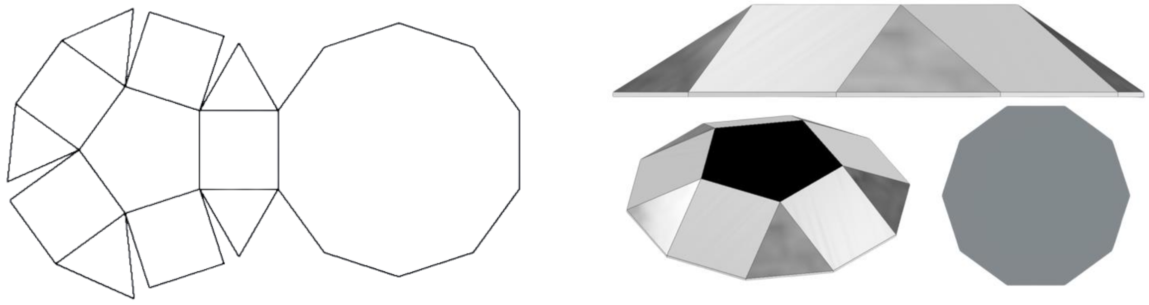
1. Introduction

There are a multitude of polyhedra that have been described over time and that are part of the construction of certain structures, objects or mechanical parts, each having particularities related to the typology of geometric two-dimensional shapes used as their basis (triangle, square, pentagon, hexagon, octagon, etc.) or the angle formed by the intersection of these geometric shapes at the vertex which can determine whether a polyhedron is convex or not. These polyhedra were categorized according to their geometric function, but also according to the personalities who contributed to the foundations of geometry as we know it today, thus, this is how Archimedean polyhedra [1], Platonic polyhedra or Johnson polyhedra appeared.

The polyhedra that are part of the category of Johnson polyhedra are geometries that contain a combination of regular polygonal faces, with the condition that the polyhedron is convex, so it does not imply the use of the same polygons as faces of the polyhedron like the Platonic polyhedra (tetrahedron, cube, icosahedron, dodecahedron etc.) [2]. Norman Johnson described a series of 92 convex polyhedra with regular polygonal faces dividing them into families and categories according to geometric characteristics (pyramids, cupola and rotundas) [3, 4]. The vast majority of the polyhedra described by Johnson are generated together with Archimedean, Platonic polyhedra or by using antiprisms or prisms, but they are formed based on precise descriptive formulas.

Depending on their shape, the cupola can serve in the construction of various objects or structures, whether is the cupola of a building, or urban furniture for children's play spaces, decorative objects or even their integration into the structure of geodesic domes.

One such geometry that was studied in this article is the pentagonal cupola (J_5), which is part of the category of Johnson's solids and is built from a pentagon, a decagon and five equilateral triangles, respectively five squares [3]. We can also obtain the pentagonal cupola by sectioning a rhombicosidodecahedron, an Archimedean solid.



a. **b.**

Figure 1. Pentagonal cupola. **a.** Pentagonal cupola net; **b.** Views of the pentagonal cupola.

The pentagonal cupola contains 15 vertices and 25 edges, that can be characterized according to the sphere radius in which it can be circumscribed, and the edge length used in polyhedron construction. Thus, the following formulas are proposed for the pentagonal cupola [5, 6]:

- Circumscribing radius of the pentagonal cupola:

$$R = \left(\frac{1}{2}\sqrt{11 + 4\sqrt{5}}\right) \cdot l \ \approx 2.2329 \cdot l \tag{1}$$

Where l is the edge length of the polygons.

- Pentagonal cupola height:

$$h = \sqrt{\frac{5 - \sqrt{5}}{10}} \cdot l \cong 0.5257 \cdot l \tag{2}$$

- Pentagonal cupola surface area:

$$A = \left(\frac{1}{4} \cdot \left(20 + 5\sqrt{3} + \sqrt{5\left(145 + 62\sqrt{5}\right)}\right)\right) \cdot l^2 \cong 16.5797 \cdot l^2 \tag{3}$$

- The volume of the pentagonal cupola:

$$V = \left(\frac{1}{6} \cdot \left(5 + 4\sqrt{5}\right)\right) \cdot l^3 \cong 2.324 \cdot l^3 \tag{4}$$

Thus, using the formulas described above, geometric characterizations of a pentagonal cupola can be determined, which, we can later integrate, for example, into a geodesic dome type structure, and can be also analysed in a virtual environment using simulations such as finite element analysis to determine the weak points of the structure [7].

2. Methods used

In this study, it was decided to use a small-scale pentagonal cupola that can later be manufactured using 3D printing equipment that provides printing beds of relatively small dimensions.

An edge of 50 mm was used for the polygons that are part of the pentagonal cupola construction. With the help of this known size of the regular polygon edges, the following was determined: the circumscription radius of ~ 111.6475 mm using formula (*1*); cupola height of ~ 26.2865 mm using formula (*2*); the pentagonal cupola volume of ~ 290506.25 mm^3 using formula (*4*). This information is useful in designing the pentagonal cupola and verifying its geometry.

2.1. Geometry generation

The geometry was generated with the help of Autodesk Inventor 2023 software, that allows the generation of solid geometric models, and which contains an analysis module of CADs by performing simulations based on the finite element method.

To design the polyhedron, in this case the decagon base was created first with a side of 50 mm, then the height of the cupola was determined, creating a plane located at the calculated distance from the base and sketching inside a pentagon with a side of 50 mm. One by one, the five triangular faces and the five square faces of the cupola were built. Ten variants of the pentagonal cupola were made, the difference between them being strictly related to the way of filling its volume. These are:

- Solid pentagonal cupola with full volume (*Figure 2* ***a.***);
- Hollow pentagonal cupola with a face thickness of 0.3 mm (*Figure 2* ***b.***);
- The pentagonal cupola filled with a circular pattern, starting from a circle with a diameter of 3 mm, the radius of the pattern increasing progressively and being located at 0.5 mm distance from each other, with a pattern wall thickness of 0.5 mm (*Figure 2* ***c.***);
- The pentagonal cupola with an ellipsoidal pattern, starting from an ellipse with a large diameter of 3 mm, the pattern increasing progressively and located at 0.5 mm from another, with a pattern thickness of 0.5 mm (*Figure 2* ***d.***);
- The pentagonal cupola filled with a slot-type pattern, starting from a slot with a length of 5 mm and width of 3 mm, the dimensions increasing progressively, with the distance between the pattern and wall thickness of 0.5 mm (*Figure 2* ***e.***);
- The pentagonal cupola with a triangular pattern, starting from an equilateral triangle, having a side of 3 mm and respecting the distance between the pattern and its thickness of 0.5 mm (*Figure 2* ***f.***);
- Pentagonal cupola with a rectangular pattern, starting from a square with a 3 mm side, a wall thickness of 0.5 mm and an equidistant distance of 0.5 mm between the inside structures (*Figure 2* ***g.***);
- The pentagonal cupola with a pentagonal pattern, starting from a pentagon, with a side of ~ 1.85 mm and a distance between the inside structures of 3 mm, respecting the *offset rules* of the pattern mentioned previously (*Figure 2* ***h.***);
- The pentagonal cupola with a hexagonal pattern, starting from a hexagon with a side of ~ 1.73 mm and with one of the distances between the sides of 3 mm, respecting the *offset rules* between the inside structures described above (*Figure 2* ***i.***);
- The pentagonal cupola with a decagon pattern, based on a decagon with a 0.97 mm side and the distance between the sides of 3 mm, respecting the *offset rules* described above in the other CAD models (*Figure 2* ***j.***).

The dimensions of the inside structure polygons and the distance between them were chosen in such a way as to ensure a relatively uniform layout of the model in each case and trying to ensure a uniform distribution of the filling percentage for each geometric model.

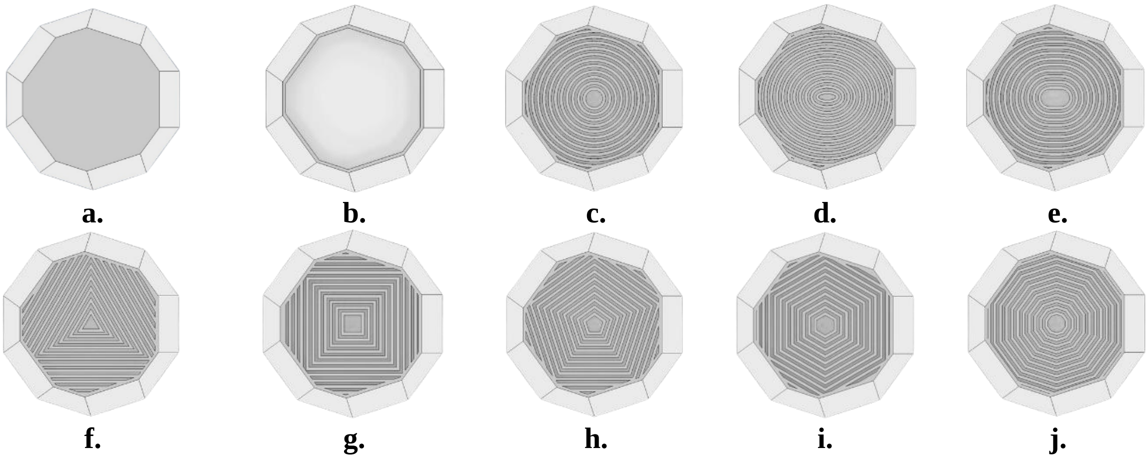

a. **b.** **c.** **d.** **e.**
f. **g.** **h.** **i.** **j.**

Figure 2. Pentagonal cupola filling pattern. **a.** Solid cupola; **b.** Empty cupola; **c.** Circular pattern; **d.** Ellipsoidal pattern; **e.** Slot-type pattern; **f.** Triangular pattern; **g.** Rectangular pattern; **h.** Pentagonal pattern; **i.** Hexagonal pattern; **j.** Decagonal pattern.

2.2. The material used in the FEA study

For this study, the use of ABS (Acrylonitrile Butadiene Styrene) [8]plastic material was chosen to use due to its popularity among additive manufacturing that uses the FDM (Fused Deposition Modeling) method, being an easily accessible material and presenting a relatively low cost of purchase [9]. It has a wide field of applicability, such as various components of household appliances, decorative elements from the automotive industry, children's toys, or various decorative elements, even interior furniture [10]. The properties of the ABS used in the static simulations carried out in this study can be observed in *Table 1*. It should also be specified that in this study, the material has a linear, isotropic, and homogeneous behavior.

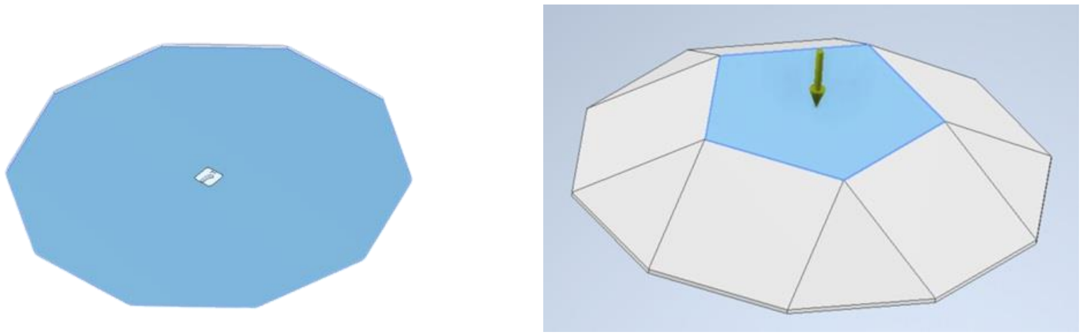
Table 1. ABS plastic material properties [*Autodesk Inventor Material Library*]

	Mechanical Properties
Young's Modulus	2.240 GPa
Poisson's Ratio	0.38
Shear Modulus	805 MPa
Density	1.060 g/cm^3
	Strength Properties
Yield Strength	20 MPa
Tensile Strength	29.6 MPa

2.3. FEA boundary conditions

For the static simulations of all 3D pentagonal cupola models, the following boundary conditions were established:

- *Fixture*: the base of the pentagonal cupola was fixed (the decagon face, *Figure 3* ***a.***).
- *Applied force*: two typologies of forces, one of 500N and one of 2000N, were applied to all generated CAD models to study the change in the mechanical behavior in both cases for all geometric models. The force was applied perpendicular to the pentagonal face of the cupola, being uniformly distributed over its entire surface (*Figure 3* ***b.***).



a. **b.**

Figure 3. Static simulation boundary conditions. **a.** Pentagonal cupola fixture; **b.** Direction and face of the applied force.

For the mesh was used an average element size (fraction of model diameter) of 0.1, a minimum element size (fraction of average size) of 0.2, a grading factor of 1.5, a maximum turn angle of 60°, summarizing a high-quality mesh of the models used in this study.

3. Results

Following the simulation to which, a force of 500 N was applied, two predictable results can be observed (*Table 2*), the solid pentagonal cupola which, presents the lowest Von Mises Stress value, and the CAD model of the empty pentagonal cupola, with walls of 0.3 mm having the highest Von Mises Stress value among all models. Analyzing the other patterns between these two extreme results, the model with the infill pattern based on the ellipses had the following Von Mises Stress value after the solid pentagonal cupola, followed by the triangle infill pattern, then the pentagon, hexagon pattern, circle pattern, decagon pattern, square and slot-type pattern.

The displacement results can be classified starting from the lowest value obtained during the application of the 500 N force (*Table 2*), as follows: solid model, then infill pattern ellipse, triangle pattern, pentagon pattern, circle pattern, hexagon pattern, decagon pattern, square pattern, slot-type pattern and finally the empty pentagonal cupola CAD model.

After analyzing the equivalent strain results, the following classification was obtained (*Table 2*), starting from the lowest value: solid model, ellipse pattern, triangle pattern, pentagon pattern, hexagon pattern, circle pattern, decagon pattern, square pattern, slot-type pattern and the empty model of the pentagonal cupola.

In *Figure 4* we can observe the graphic representation of the static simulation results following the application of 500 N force on each analyzed pentagonal cupola CAD model.

Regarding the results of the static simulation of 2000 N force applied, the pentagonal cupola models tend to follow the same behavior (*Table 3*), as expected, the results of the values increased considerably in all cases by approximately 4 times compared to the simulations performed with the applied force of 500 N.

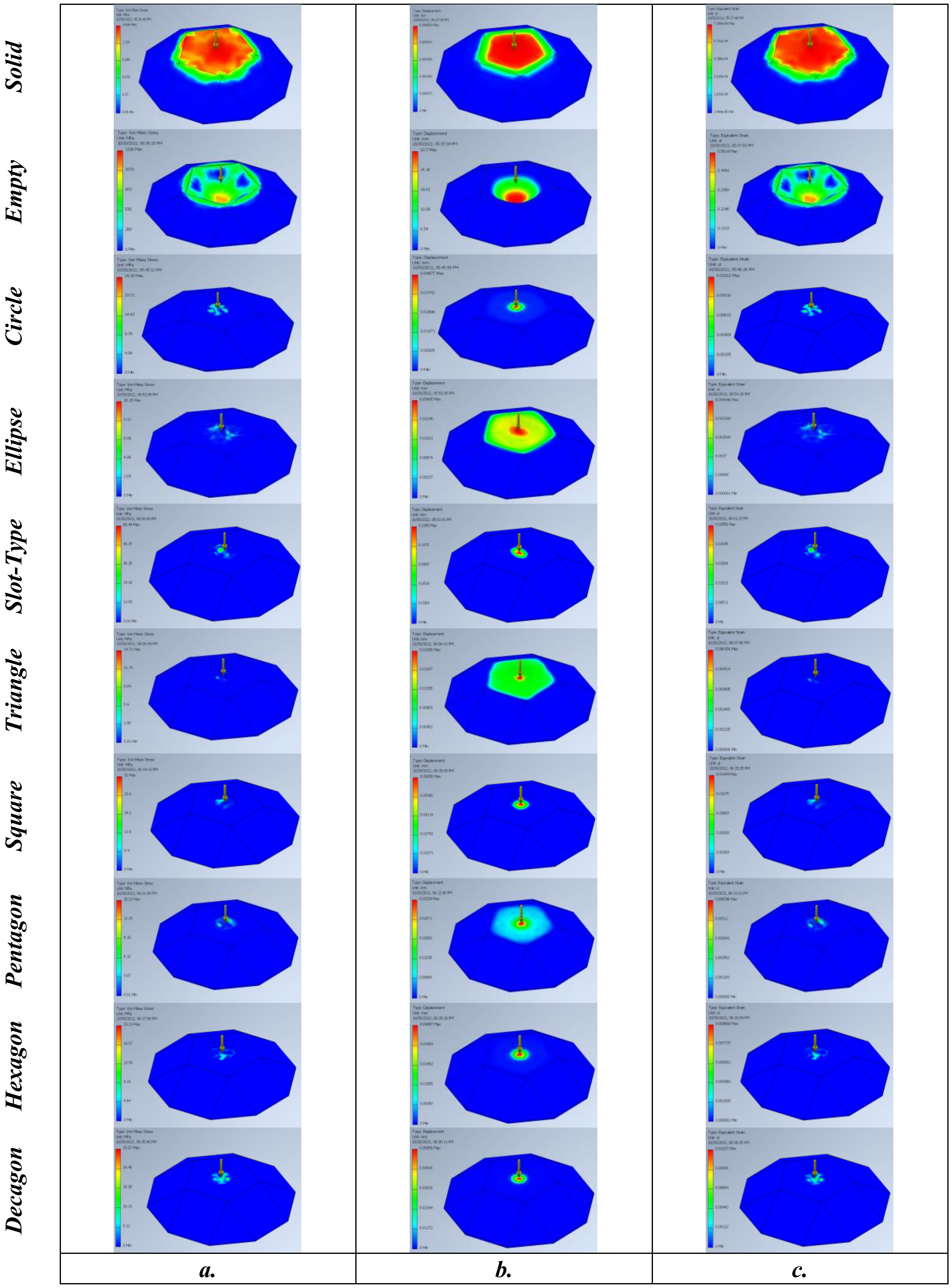


Figure 4. Results of static simulation for 500 N load applied. **a.** Von Mises Stress results for all types of infill pattern; **b.** Displacement results for all types of infill pattern; **c.** Equivalent Strain results for all types of infill pattern.

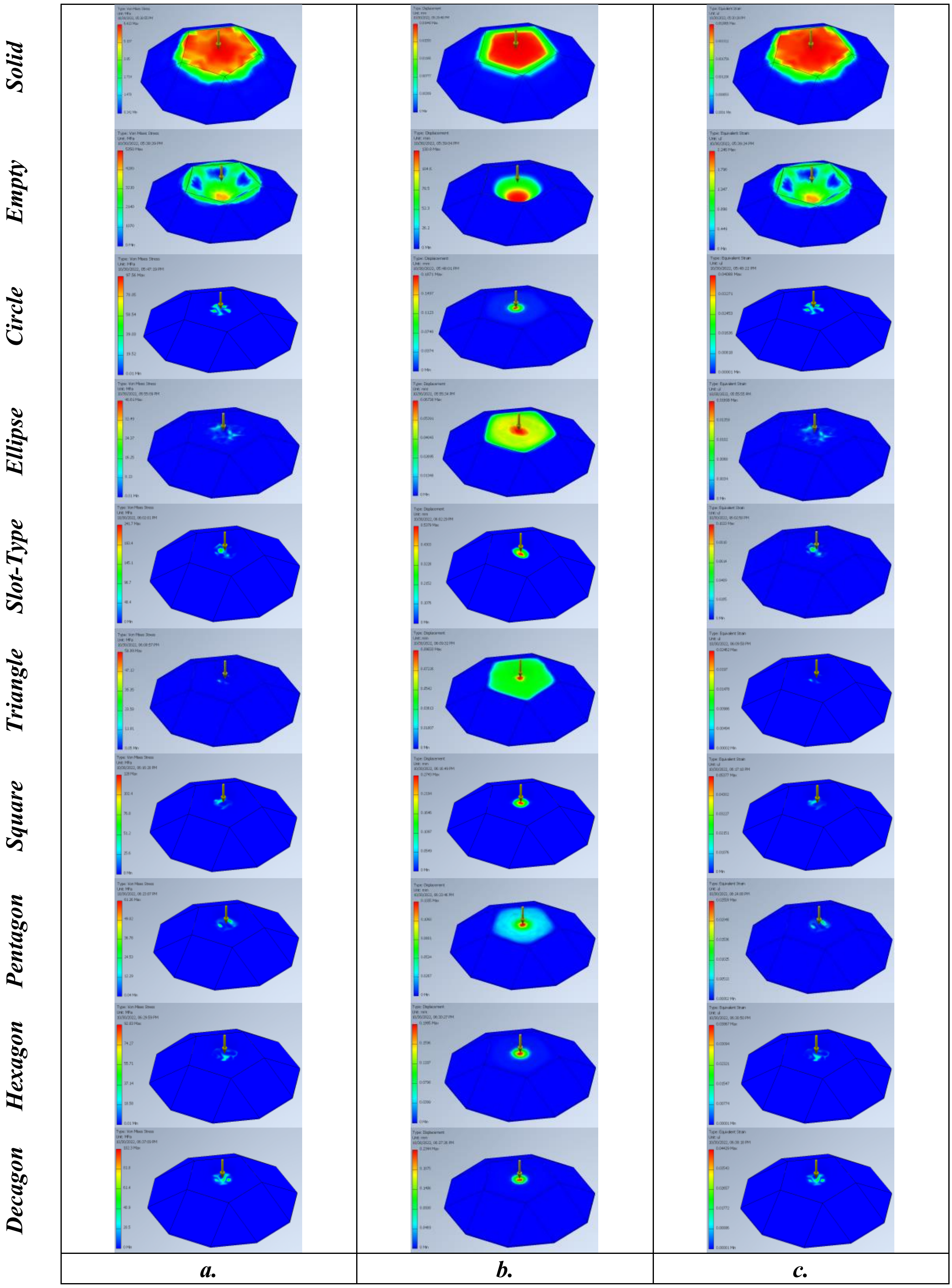


Figure 5. Results of static simulation for 2000 N load applied. **a.** Von Mises Stress results for all types of infill pattern; **b.** Displacement results for all types of infill pattern; **c.** Equivalent Strain results for all types of infill pattern.

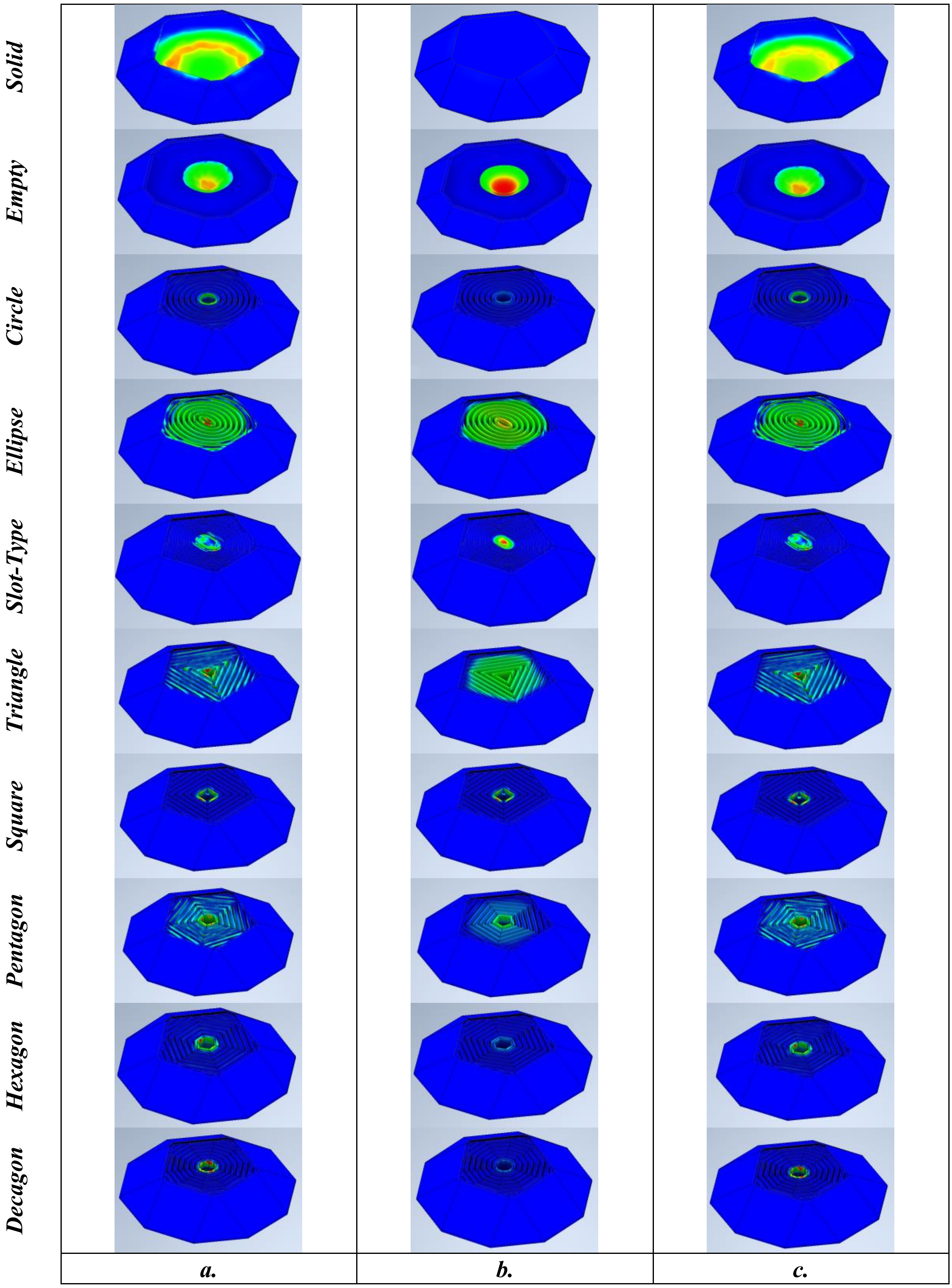


Figure 6. Sectioned pentagonal cupola static results simulation for 2000 N load applied. **a.** Von Mises Stress results for all types of infill pattern; **b.** Displacement results for all types of infill pattern; **c.** Equivalent Strain results for all types of infill pattern.

Table 2. Results of static simulation for 500 N load applied in case of all infill patterns

Pattern	*Von Mises Stress [MPa]*		*Displacement [mm]*		*Equivalent Strain [ul]*	
	Minimum	*Maximum*	*Minimum*	*Maximum*	*Minimum*	*Maximum*
Solid	0.06	1.606	0	0.004858	$2.499 \cdot 10^{-5}$	$7.164 \cdot 10^{-4}$
Empty	0.00	1338	0	32.70000	0.000000	0.561400
Circle	0.00	24.39	0	0.046770	0.000000	0.010220
Ellipse	0.00	10.15	0	0.016850	0.000001	0.004248
Slot-Type	0.01	60.44	0	0.134500	0.000000	0.025560
Triangle	0.01	14.72	0	0.022580	0.000006	0.006154
Square	0.00	32.00	0	0.068560	0.000000	0.013440
Pentagon	0.01	15.32	0	0.033390	0.000005	0.006398
Hexagon	0.00	23.21	0	0.049870	0.000002	0.009668
Decagon	0.00	25.57	0	0.058590	0.000000	0.011070

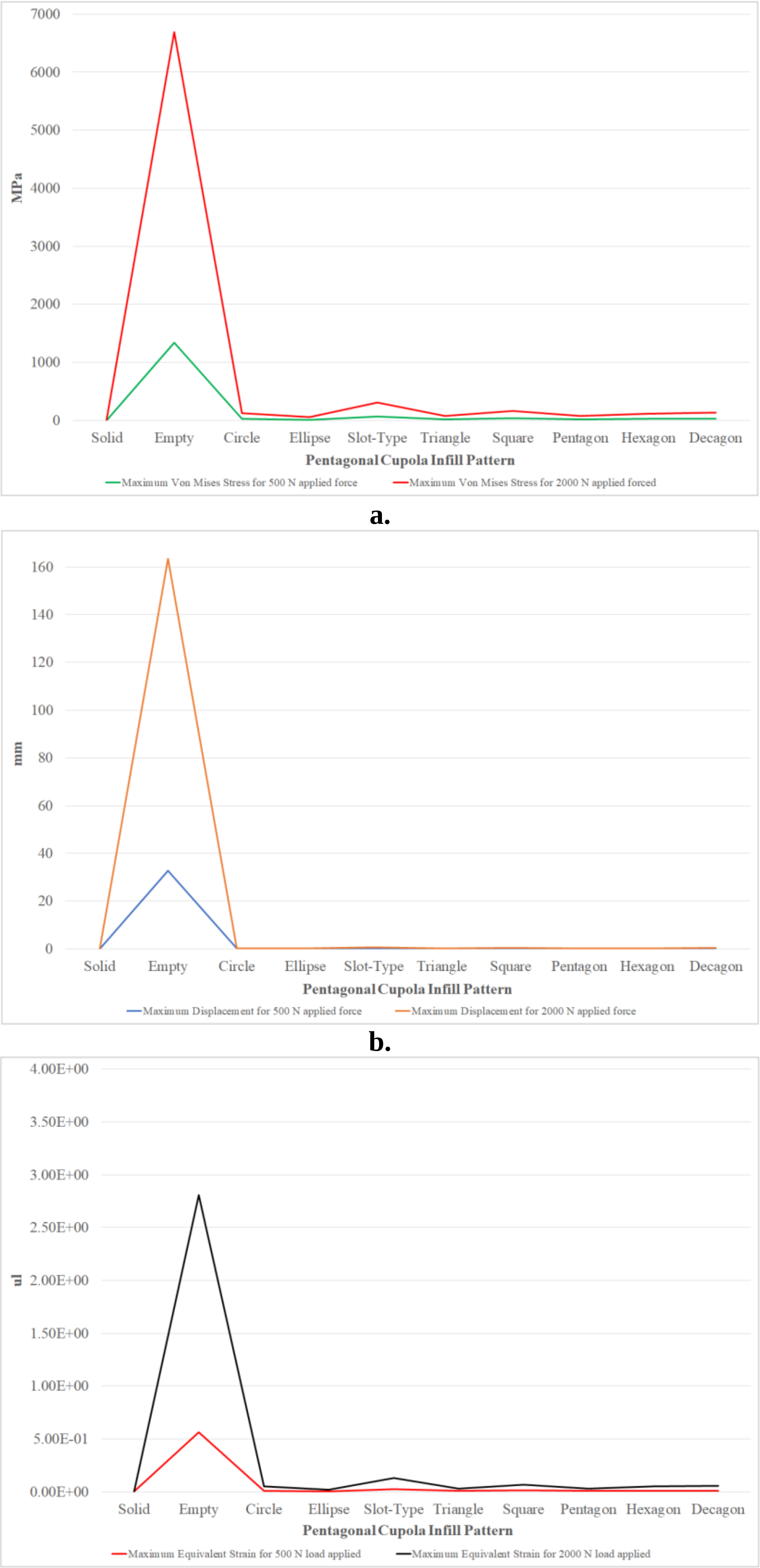
In *Figure 6,* the CAD model sections are shown to visualize the simulation results for the 2000 N force applied on the pentagonal cupola.

In *Figure 7* ***a.*** the graphic evolution of Von Mises Stress results is presented in case of both application forces, 500 N and 2000 N. The displacement results can be visualized in a graphic form in *Figure 7* **b.** and the equivalent strain results are represented graphically in *Figure 7* ***c.***

The biggest result gab can be seen at the empty CAD model of the pentagonal cupola, which, is built only by the simple constant wall thickness of 0.3 mm, thus obtaining very high values of Von Mises Stress results, and some displacements beyond the limit.

Table 3. Results of static simulation for 2000 N load applied in case of all infill patterns

Pattern	*Von Mises Stress [MPa]*		*Displacement [mm]*		*Equivalent Strain [ul]*	
	Minimum	*Maximum*	*Minimum*	*Maximum*	*Minimum*	*Maximum*
Solid	0.242	6.423	0	0.01943	0.00010	0.002866
Empty	0.00	5350	0	130.800	0.00000	2.246000
Circle	0.01	97.56	0	0.18710	0.00001	0.040880
Ellipse	0.01	40.61	0	0.06738	0.00000	0.016990
Slot-Type	0.00	241.7	0	0.53790	0.00000	0.102300
Triangle	0.05	58.89	0	0.09033	0.00002	0.024620
Square	0.00	128.0	0	0.27430	0.00000	0.053770
Pentagon	0.04	61.26	0	0.13350	0.00002	0.025590
Hexagon	0.01	92.83	0	0.19950	0.00001	0.038670
Decagon	0.00	102.3	0	0.23440	0.00001	0.044290



a.

b.

c.

Figure 7. Graphic representation of the static simulation results for both loads **a.** Von Mises Stress results; **b.** Displacement results; **c.** Equivalent Strain results.

4. Conclusions

Although we often do not give importance, polyhedra and intersections of polyhedra are an integral part of our lives in different forms, either as objects that surround us with a decorative or functional role, or as metal structures like geodesic domes and not only. Observing the mechanical behaviour of the pentagonal cupola in case of different filling patterns, gives us information that we can use, especially when we choose this geometric shape to be manufactured by using 3D printing. Depending on the field and the purpose of the designed shape, we can choose the infill pattern by considering the forces to which it can be subjected.

Although the results of the static simulation were predictable regarding the solid model of the pentagonal cupola by showing the best results in the case of both applied forces, a possible option if we want to use less material, can be the ellipsoid pattern, but is worth noting that the triangular and pentagonal filling patterns offered also better results compared to the other filling patterns used in this study.

The choice of using a filling pattern in the case of additive manufacturing and a filling percentage as low as possible reduces the amount of material used in the realization of a shape, the printing time and implicitly the energy used to manufacture the part, but at the same time it decreases its resistance at external loads.

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